11.1 INTRODUCTION

Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this chapter is power analysis.

Power analysis is of paramount importance. Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device—every fan, motor, lamp, pressing iron, TV, personal computer—has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance. The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer.

We will begin by defining and deriving *instantaneous power* and *average power*. We will then introduce other power concepts. As practical applications of these concepts, we will discuss how power is measured and reconsider how electric utility companies charge their customers.

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11.2 INSTANTANEOUS AND AVERAGE POWER

As mentioned in Chapter 2, the *instantaneous power* $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention,

$$p(t) = v(t)i(t) \tag{11.1}$$

The instantaneous power is the power at any instant of time. It is the rate at which an element absorbs energy.

We can also think of the instantaneous power as the power absorbed by the element at a specific instant of time. Instantaneous quantities are denoted by lowercase letters.

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 11.1. Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{11.2a}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{11.2b}$$

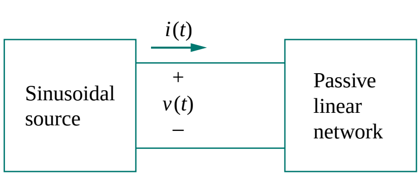

Figure 11.1 Sinusoidal source and passive linear circuit.

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \tag{11.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{11.4}$$

and express Eq. (11.3) as

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i) \tag{11.5}$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

A sketch of $p(t)$ in Eq. (11.5) is shown in Fig. 11.2, where $T = 2\pi/\omega$ is the period of voltage or current. We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. We also observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

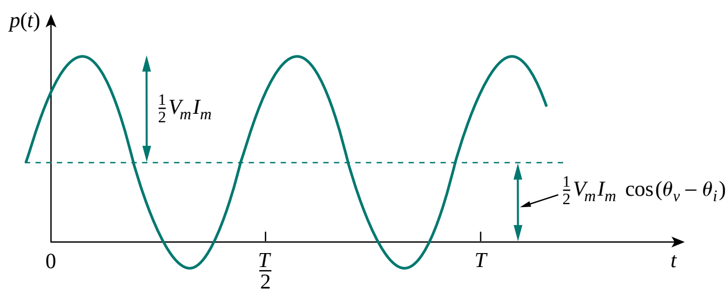

Figure 11.2 The instantaneous power $p(t)$ entering a circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power.

The average power is the average of the instantaneous power over one period.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt \tag{11.6}$$

Although Eq. (11.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period of $p(t)$ which is $T_0 = T/2$.

Substituting $p(t)$ in Eq. (11.5) into Eq. (11.6) gives

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T} \int_0^T dt + \frac{1}{2} V_m I_m \frac{1}{T} \int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt \qquad (11.7)$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (11.7) vanishes and the average power becomes

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \qquad (11.8)$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while P does not depend on time. To find the instantaneous power, we must necessarily have $v(t)$ and $i(t)$ in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (11.2), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (11.2) are $\mathbf{V} = V_m \angle \theta_v$ and $\mathbf{I} = I_m \angle \theta_i$, respectively. P is calculated using Eq. (11.8) or using phasors $\mathbf{V}$ and $\mathbf{I}$. To use phasors, we notice that

$$\begin{aligned} \frac{1}{2}\mathbf{V}\mathbf{I}^* &= \frac{1}{2} V_m I_m \angle \theta_v - \theta_i \\ &= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)] \end{aligned} \qquad (11.9)$$

We recognize the real part of this expression as the average power P according to Eq. (11.8). Thus,

$$P = \frac{1}{2} \operatorname{Re}\left[\mathbf{V}\mathbf{I}^*\right] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \qquad (11.10)$$

Consider two special cases of Eq. (11.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R \qquad (11.11)$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. Equation (11.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90^\circ$, we have a purely reactive circuit, and

$$P = \frac{1}{2} V_m I_m \cos 90^\circ = 0 \qquad (11.12)$$

showing that a purely reactive circuit absorbs no average power. In summary,

> A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

EXAMPLE 11.1

Given that

$$v(t) = 120\cos(377t + 45^\circ)\ \text{V} \quad \text{and} \quad i(t) = 10\cos(377t - 10^\circ)\ \text{A}$$

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A\cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

PRACTICE PROBLEM 11.1

Calculate the instantaneous power and average power absorbed by the passive linear network of Fig. 11.1 if

$$v(t) = 80\cos(10t + 20^\circ)\ \text{V} \quad \text{and} \quad i(t) = 15\sin(10t + 60^\circ)\ \text{A}$$

Answer: $385.7 + 600\cos(20t - 10^\circ)$ W, 385.7 W.

EXAMPLE 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0^\circ$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0^\circ}{30 - j70} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} = 1.576\angle 66.8^\circ \text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24 \text{ W}$$

PRACTICE PROBLEM 11.2

A current $\mathbf{I} = 10\angle 30^\circ$ flows through an impedance $\mathbf{Z} = 20\angle -22^\circ\ \Omega$. Find the average power delivered to the impedance.

Answer: 927.2 W.

EXAMPLE 11.3

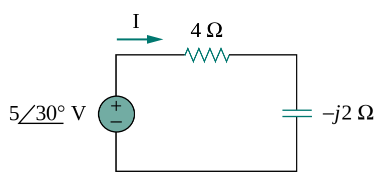

Figure 11.3 For Example 11.3.

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

Solution:

The current $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I} = \mathbf{I}_R = 1.118\angle 56.57^\circ \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

PRACTICE PROBLEM 11.3

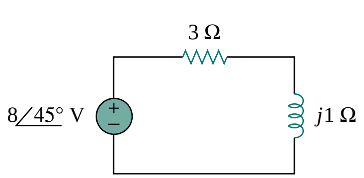

Figure 11.4 For Practice Prob. 11.3.

In the circuit of Fig. 11.4, calculate the average power absorbed by the resistor and inductor. Find the average power supplied by the voltage source.

Answer: 9.6 W, 0 W, 9.6 W.

EXAMPLE 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

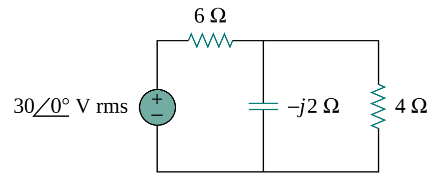

Figure 11.18 For Example 11.10.

Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\underline{/-13.24^\circ}\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\underline{/0^\circ}}{7\underline{/-13.24^\circ}} = 4.286\underline{/13.24^\circ}\ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\ \text{pf} = (30)(4.286)0.9734 = 125\ \text{W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125\ \text{W}$$

where R is the resistive part of $\mathbf{Z}$.

PRACTICE PROBLEM 11.10

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

Answer: 0.936 lagging, 118 W.

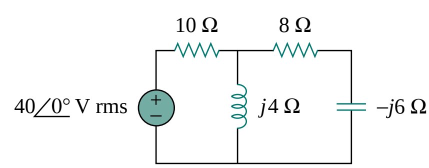

Figure 11.19 For Practice Prob. 11.10.

11.6 COMPLEX POWER

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Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load.

Consider the ac load in Fig. 11.20. Given the phasor form $\mathbf{V} = V_m\underline{/\theta_v}$ and $\mathbf{I} = I_m\underline{/\theta_i}$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{11.40}$$

assuming the passive sign convention (see Fig. 11.20). In terms of the rms values,

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* \tag{11.41}$$

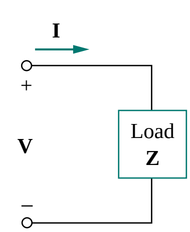

Figure 11.20 The voltage and current phasors associated with a load.

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}} \angle \theta_v \tag{11.42}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}} \angle \theta_i \tag{11.43}$$

When working with the rms values of currents or voltages, we may drop the subscript rms if no confusion will be caused by doing so.

Thus we may write Eq. (11.41) as

$$\begin{aligned} \mathbf{S} &= V_{\text{rms}} I_{\text{rms}} \angle \theta_v - \theta_i \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) + j V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \end{aligned} \tag{11.44}$$

This equation can also be obtained from Eq. (11.9). We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

The complex power may be expressed in terms of the load impedance **Z**. From Eq. (11.37), the load impedance **Z** may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i \tag{11.45}$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$. Substituting this into Eq. (11.41) gives

$$\boxed{\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*}} \tag{11.46}$$

Since $\mathbf{Z} = R + jX$, Eq. (11.46) becomes

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ \tag{11.47}$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \tag{11.48}$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \tag{11.49}$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

Comparing Eq. (11.44) with Eq. (11.47), we notice that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \tag{11.50}$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

> **Complex power** (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\begin{aligned} \text{Complex Power} &= \mathbf{S} = P + jQ = \frac{1}{2}\mathbf{VI}^* \\ &= V_{\text{rms}} I_{\text{rms}} \underline{/\theta_v - \theta_i} \\ \text{Apparent Power} &= S = |\mathbf{S}| = V_{\text{rms}} I_{\text{rms}} = \sqrt{P^2 + Q^2} \\ \text{Real Power} &= P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\ \text{Reactive Power} &= Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\ \text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i) \end{aligned} \tag{11.51}$$

This shows how the complex power contains *all* the relevant power information in a given load.

S contains *all* power information of a load. The real part of **S** is the real power P; its imaginary part is the reactive power Q; its magnitude is the apparent power S; and the cosine of its phase angle is the power factor pf.

It is a standard practice to represent S, P, and Q in the form of a triangle, known as the *power triangle*, shown in Fig. 11.21(a). This is similar to the impedance triangle showing the relationship between $\mathbf{Z}$, R, and X, illustrated in Fig. 11.21(b). The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 11.22, when **S** lies in the first quadrant, we have an inductive load and a lagging pf. When **S** lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

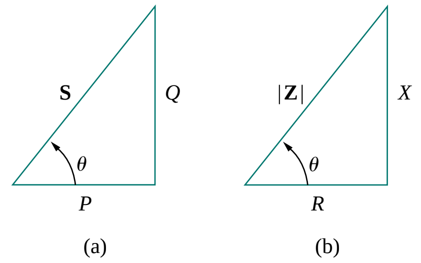

Figure 11.21 (a) Power triangle, (b) impedance triangle.

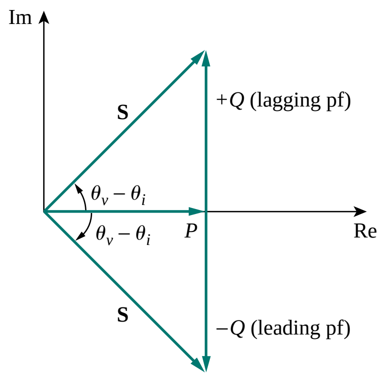

Figure 11.22 Power triangle.

EXAMPLE 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\angle -10^\circ, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\angle +50^\circ$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = \left(\frac{60}{\sqrt{2}}\angle -10^\circ\right)\left(\frac{1.5}{\sqrt{2}}\angle -50^\circ\right) = 45\angle -60^\circ \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle -60^\circ = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^\circ}{1.5\angle +50^\circ} = 40\angle -60^\circ\ \Omega$$

which is a capacitive impedance.

PRACTICE PROBLEM 11.11

For a load, $\mathbf{V}_{\text{rms}} = 110\angle 85^\circ$ V, $\mathbf{I}_{\text{rms}} = 0.4\angle 15^\circ$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\angle 70^\circ$ VA, 44 VA, (b) 15.05 W, 41.35 VAR,
(c) 0.342 lagging, $94.06 + j258.4\ \Omega$.

EXAMPLE 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf $= \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\angle 0^\circ} = 85.6 + j51.7 \text{ A} = 100\angle 31.13^\circ \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\angle -31.13^\circ$ and the peak current is

$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{120\angle 0^\circ}{100\angle -31.13^\circ} = 1.2\angle 31.13^\circ \ \Omega$$

which is an inductive impedance.

PRACTICE PROBLEM 11.12

A sinusoidal source supplies 10 kVA reactive power to load $\mathbf{Z} = 250\angle -75^\circ \ \Omega$. Determine: (a) the power factor, (b) the apparent power delivered to the load, and (c) the peak voltage.

Answer: (a) 0.2588 leading, (b) −10.35 kVAR, (c) 2.275 kV.

†11.7 CONSERVATION OF AC POWER

The principle of conservation of power applies to ac circuits as well as to dc circuits (see Section 1.5).

In fact, we already saw in Examples 11.3 and 11.4 that average power is conserved in ac circuits.

To see this, consider the circuit in Fig. 11.23(a), where two load impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are connected in parallel across an ac source $\mathbf{V}$. KCL gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 \tag{11.52}$$

The complex power supplied by the source is

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}\mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \frac{1}{2}\mathbf{V}\mathbf{I}_1^* + \frac{1}{2}\mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.53}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

amount of wire required for a three-phase system is less than that required for an equivalent single-phase system.

We begin with a discussion of balanced three-phase voltages. Then we analyze each of the four possible configurations of balanced three-phase systems. We also discuss the analysis of unbalanced three-phase systems. We learn how to use *PSpice for Windows* to analyze a balanced or unbalanced three-phase system. Finally, we apply the concepts developed in this chapter to three-phase power measurement and residential electrical wiring.

12.2 BALANCED THREE-PHASE VOLTAGES

Three-phase voltages are often produced with a three-phase ac generator (or alternator) whose cross-sectional view is shown in Fig. 12.4. The generator basically consists of a rotating magnet (called the *rotor*) surrounded by a stationary winding (called the *stator*). Three separate windings or coils with terminals a-a', b-b', and c-c' are physically placed 120° apart around the stator. Terminals a and a', for example, stand for one of the ends of coils going into and the other end coming out of the page. As the rotor rotates, its magnetic field "cuts" the flux from the three coils and induces voltages in the coils. Because the coils are placed 120° apart, the induced voltages in the coils are equal in magnitude but out of phase by 120° (Fig. 12.5). Since each coil can be regarded as a single-phase generator by itself, the three-phase generator can supply power to both single-phase and three-phase loads.

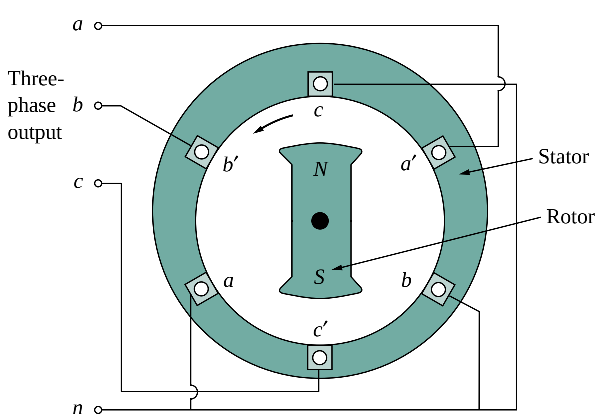

Figure 12.4 A three-phase generator.

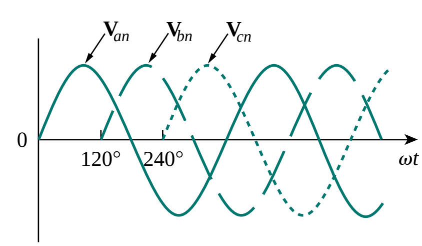

Figure 12.5 The generated voltages are 120° apart from each other.

A typical three-phase system consists of three voltage sources connected to loads by three or four wires (or transmission lines). (Three-phase current sources are very scarce.) A three-phase system is equivalent to three single-phase circuits. The voltage sources can be either wye-connected as shown in Fig. 12.6(a) or delta-connected as in Fig. 12.6(b).

Let us consider the wye-connected voltages in Fig. 12.6(a) for now. The voltages $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are respectively between lines a, b, and

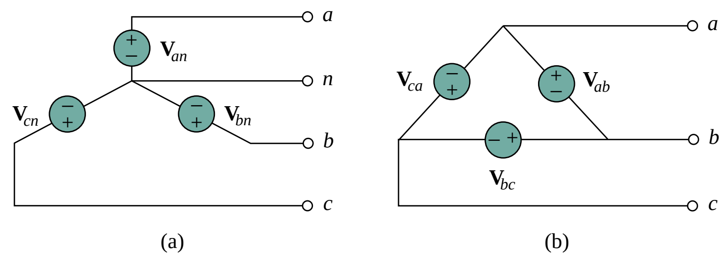

Figure 12.6 Three-phase voltage sources: (a) Y-connected source, (b) Δ-connected source.

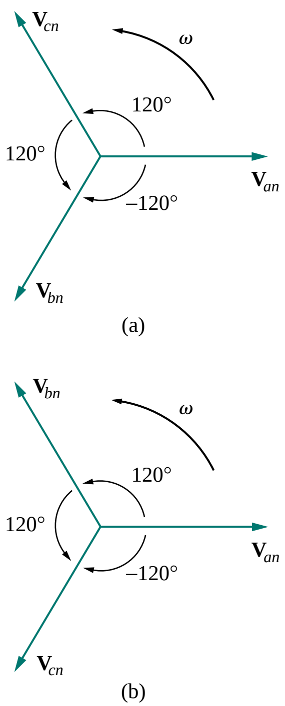

Figure 12.7 Phase sequences: (a) *abc* or positive sequence, (b) *acb* or negative sequence.

As a common tradition in power systems, voltage and current in this chapter are in rms values unless otherwise stated.

c, and the neutral line n. These voltages are called *phase voltages*. If the voltage sources have the same amplitude and frequency ω and are out of phase with each other by 120°, the voltages are said to be *balanced*. This implies that

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0 \tag{12.1}$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \tag{12.2}$$

Thus,

> Balanced phase voltages are equal in magnitude and are out of phase with each other by 120°.

Since the three-phase voltages are 120° out of phase with each other, there are two possible combinations. One possibility is shown in Fig. 12.7(a) and expressed mathematically as

$$\begin{aligned}\mathbf{V}_{an} &= V_p\underline{/0^\circ}\\ \mathbf{V}_{bn} &= V_p\underline{/-120^\circ}\\ \mathbf{V}_{cn} &= V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ}\end{aligned} \tag{12.3}$$

where V_p is the effective or rms value. This is known as the *abc sequence* or *positive sequence*. In this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$, which in turn leads $\mathbf{V}_{cn}$. This sequence is produced when the rotor in Fig. 12.4 rotates counterclockwise. The other possibility is shown in Fig. 12.7(b) and is given by

$$\begin{aligned}\mathbf{V}_{an} &= V_p\underline{/0^\circ}\\ \mathbf{V}_{cn} &= V_p\underline{/-120^\circ}\\ \mathbf{V}_{bn} &= V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ}\end{aligned} \tag{12.4}$$

This is called the *acb sequence* or *negative sequence*. For this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$, which in turn leads $\mathbf{V}_{bn}$. The *acb* sequence is produced when the rotor in Fig. 12.4 rotates in the clockwise direction.

It is easy to show that the voltages in Eqs. (12.3) or (12.4) satisfy Eqs. (12.1) and (12.2). For example, from Eq. (12.3),

$$\begin{aligned}\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} &= V_p\underline{/0^\circ} + V_p\underline{/-120^\circ} + V_p\underline{/+120^\circ} \\ &= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866) \\ &= 0\end{aligned} \tag{12.5}$$

The **phase sequence** is the time order in which the voltages pass through their respective maximum values.

The phase sequence may also be regarded as the order in which the phase voltages reach their peak (or maximum) values with respect to time.

The phase sequence is determined by the order in which the phasors pass through a fixed point in the phase diagram.

Reminder: As time increases, each phasor (or sinor) rotates at an angular velocity ω.

In Fig. 12.7(a), as the phasors rotate in the counterclockwise direction with frequency ω, they pass through the horizontal axis in a sequence *abcabca* .... Thus, the sequence is *abc* or *bca* or *cab*. Similarly, for the phasors in Fig. 12.7(b), as they rotate in the counterclockwise direction, they pass the horizontal axis in a sequence *acbacba* .... This describes the *acb* sequence. The phase sequence is important in three-phase power distribution. It determines the direction of the rotation of a motor connected to the power source, for example.

Like the generator connections, a three-phase load can be either wye-connected or delta-connected, depending on the end application. Figure 12.8(a) shows a wye-connected load, and Fig. 12.8(b) shows a delta-connected load. The neutral line in Fig. 12.8(a) may or may not be there, depending on whether the system is four- or three-wire. (And, of course, a neutral connection is topologically impossible for a delta connection.) A wye- or delta-connected load is said to be *unbalanced* if the phase impedances are not equal in magnitude or phase.

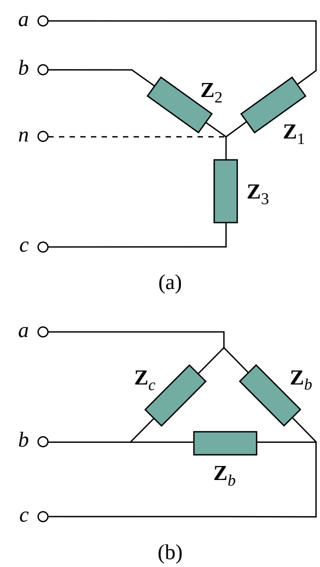

Figure 12.8 Two possible three-phase load configurations: (a) a Y-connected load, (b) a Δ-connected load

A **balanced load** is one in which the phase impedances are equal in magnitude and in phase.

For a *balanced* wye-connected load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y \tag{12.6}$$

where $\mathbf{Z}_Y$ is the load impedance per phase. For a *balanced* delta-connected load,

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta \tag{12.7}$$

Reminder: A Y-connected load consists of three impedances connected to a neutral node, while a Δ-connected load consists of three impedances connected around a loop. The load is balanced when the three impedances are equal in either case.

where $\mathbf{Z}_\Delta$ is the load impedance per phase in this case. We recall from Eq. (9.69) that

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \tag{12.8}$$

so we know that a wye-connected load can be transformed into a delta-connected load, or vice versa, using Eq. (12.8).

Since both the three-phase source and the three-phase load can be either wye- or delta-connected, we have four possible connections:

- Y-Y connection (i.e., Y-connected source with a Y-connected load).
- Y-Δ connection.
- Δ-Δ connection.
- Δ-Y connection.

In subsequent sections, we will consider each of these possible configurations.

It is appropriate to mention here that a balanced delta-connected load is more common than a balanced wye-connected load. This is due to the ease with which loads may be added or removed from each phase of a delta-connected load. This is very difficult with a wye-connected load because the neutral may not be accessible. On the other hand, delta-connected sources are not common in practice because of the circulating current that will result in the delta-mesh if the three-phase voltages are slightly unbalanced.

EXAMPLE 12.1

Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10^\circ)$$
$$v_{bn} = 200\cos(\omega t - 230^\circ), \qquad v_{cn} = 200\cos(\omega t - 110^\circ)$$

Solution:

The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\underline{/10^\circ}, \qquad \mathbf{V}_{bn} = 200\underline{/-230^\circ}, \qquad \mathbf{V}_{cn} = 200\underline{/-110^\circ}$$

We notice that $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120° and $\mathbf{V}_{cn}$ in turn leads $\mathbf{V}_{bn}$ by 120°. Hence, we have an *acb* sequence.

PRACTICE PROBLEM 12.1

Given that $\mathbf{V}_{bn} = 110\underline{/30^\circ}$, find $\mathbf{V}_{an}$ and $\mathbf{V}_{cn}$, assuming a positive (*abc*) sequence.

Answer: $110\underline{/150^\circ}$, $110\underline{/-90^\circ}$.

12.3 BALANCED WYE-WYE CONNECTION

We begin with the Y-Y system, because any balanced three-phase system can be reduced to an equivalent Y-Y system. Therefore, analysis of this system should be regarded as the key to solving all balanced three-phase systems.

> A **balanced Y-Y system** is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.

Consider the balanced four-wire Y-Y system of Fig. 12.9, where a Y-connected load is connected to a Y-connected source. We assume a balanced load so that load impedances are equal. Although the impedance $\mathbf{Z}_Y$ is the total load impedance per phase, it may also be regarded as the sum of the source impedance $\mathbf{Z}_s$, line impedance $\mathbf{Z}_\ell$, and load impedance $\mathbf{Z}_L$ for each phase, since these impedances are in series. As illustrated in Fig. 12.9, $\mathbf{Z}_s$ denotes the internal impedance of the phase winding of the generator; $\mathbf{Z}_\ell$ is the impedance of the line joining a phase of the source with a phase of the load; $\mathbf{Z}_L$ is the impedance of each phase of the load; and $\mathbf{Z}_n$ is the impedance of the neutral line. Thus, in general

$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L \tag{12.9}$$

$\mathbf{Z}_s$ and $\mathbf{Z}_\ell$ are often very small compared with $\mathbf{Z}_L$, so one can assume that $\mathbf{Z}_Y = \mathbf{Z}_L$ if no source or line impedance is given. In any event, by lumping the impedances together, the Y-Y system in Fig. 12.9 can be simplified to that shown in Fig. 12.10.

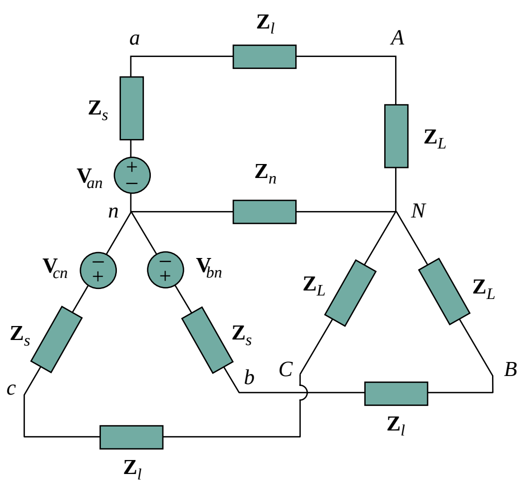

Figure 12.9 A balanced Y-Y system, showing the source, line, and load impedances.

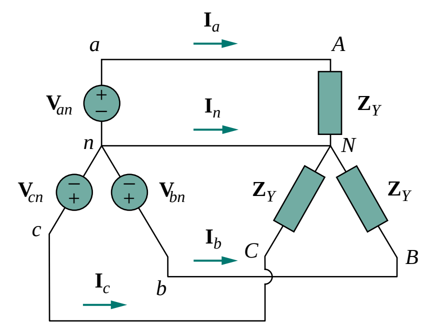

Figure 12.10 Balanced Y-Y connection.

Assuming the positive sequence, the *phase* voltages (or line-to-neutral voltages) are

$$\begin{gathered}\mathbf{V}_{an} = V_p\angle 0^\circ \\ \mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ\end{gathered} \tag{12.10}$$

The *line-to-line* voltages or simply *line* voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are related to the phase voltages. For example,

$$\begin{aligned}\mathbf{V}_{ab} &= \mathbf{V}_{an} + \mathbf{V}_{nb} = \mathbf{V}_{an} - \mathbf{V}_{bn} = V_p\angle 0^\circ - V_p\angle -120^\circ \\ &= V_p\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \sqrt{3}V_p\angle 30^\circ\end{aligned} \tag{12.11a}$$

Similarly, we can obtain

$$\mathbf{V}_{bc} = \mathbf{V}_{bn} - \mathbf{V}_{cn} = \sqrt{3}V_p\angle -90^\circ \tag{12.11b}$$

$$\mathbf{V}_{ca} = \mathbf{V}_{cn} - \mathbf{V}_{an} = \sqrt{3}V_p\angle -210^\circ \tag{12.11c}$$

Thus, the magnitude of the line voltages V_L is $\sqrt{3}$ times the magnitude of the phase voltages V_p, or

$$V_L = \sqrt{3}V_p \tag{12.12}$$

where

$$V_p = |\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \tag{12.13}$$

and

$$V_L = |\mathbf{V}_{ab}| = |\mathbf{V}_{bc}| = |\mathbf{V}_{ca}| \tag{12.14}$$

Also the line voltages lead their corresponding phase voltages by 30°. Figure 12.11(a) illustrates this. Figure 12.11(a) also shows how to determine $\mathbf{V}_{ab}$ from the phase voltages, while Fig. 12.11(b) shows the same for the three line voltages. Notice that $\mathbf{V}_{ab}$ leads $\mathbf{V}_{bc}$ by 120°, and $\mathbf{V}_{bc}$ leads $\mathbf{V}_{ca}$ by 120°, so that the line voltages sum up to zero as do the phase voltages.

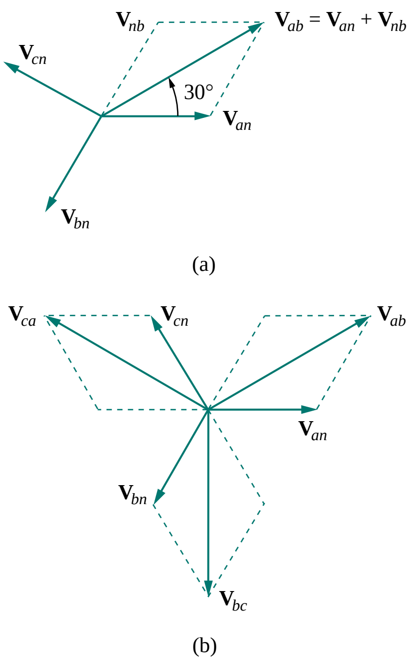

Figure 12.11 Phasor diagrams illustrating the relationship between line voltages and phase voltages.

Applying KVL to each phase in Fig. 12.10, we obtain the line currents as

$$\begin{aligned} \mathbf{I}_a &= \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle -120^\circ}{\mathbf{Z}_Y} = \mathbf{I}_a\angle -120^\circ \\ \mathbf{I}_c &= \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\angle -240^\circ}{\mathbf{Z}_Y} = \mathbf{I}_a\angle -240^\circ \end{aligned} \tag{12.15}$$

We can readily infer that the line currents add up to zero,

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0 \tag{12.16}$$

so that

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0 \tag{12.17a}$$

or

$$\mathbf{V}_{nN} = \mathbf{Z}_n\mathbf{I}_n = 0 \tag{12.17b}$$

that is, the voltage across the neutral wire is zero. The neutral line can thus be removed without affecting the system. In fact, in long distance power transmission, conductors in multiples of three are used with the earth itself acting as the neutral conductor. Power systems designed in this way are well grounded at all critical points to ensure safety.

While the *line* current is the current in each line, the *phase* current is the current in each phase of the source or load. In the Y-Y system, the line current is the same as the phase current. We will use single subscripts for line currents because it is natural and conventional to assume that line currents flow from the source to the load.

An alternative way of analyzing a balanced Y-Y system is to do so on a "per phase" basis. We look at one phase, say phase a, and analyze the single-phase equivalent circuit in Fig. 12.12. The single-phase analysis yields the line current $\mathbf{I}_a$ as

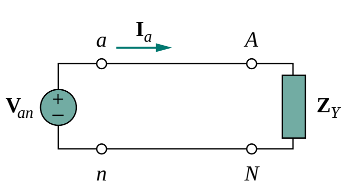

Figure 12.12 A single-phase equivalent circuit.

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} \tag{12.18}$$

From $\mathbf{I}_a$, we use the phase sequence to obtain other line currents. Thus, as long as the system is balanced, we need only analyze one phase. We may do this even if the neutral line is absent, as in the three-wire system.

EXAMPLE 12.2

Calculate the line currents in the three-wire Y-Y system of Fig. 12.13.

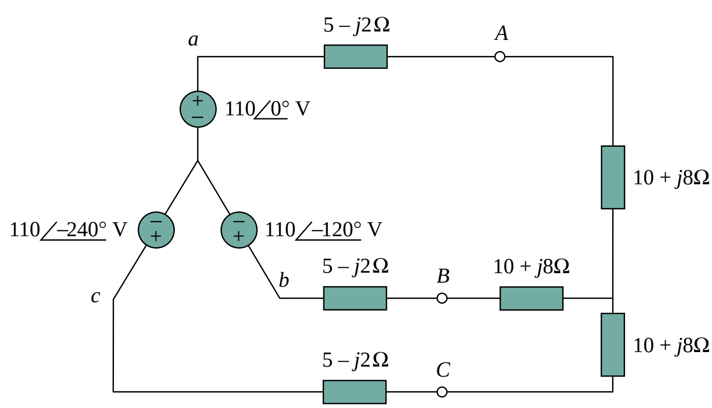

Figure 12.13 Three-wire Y-Y system; for Example 12.2.

Solution:

The three-phase circuit in Fig. 12.13 is balanced; we may replace it with its single-phase equivalent circuit such as in Fig. 12.12. We obtain $\mathbf{I}_a$ from the single-phase analysis as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

where $\mathbf{Z}_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\underline{/21.8^\circ}$. Hence,

$$\mathbf{I}_a = \frac{110\underline{/0^\circ}}{16.155\underline{/21.8^\circ}} = 6.81\underline{/-21.8^\circ}\ \text{A}$$

Since the source voltages in Fig. 12.13 are in positive sequence and the line currents are also in positive sequence,

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 6.81\underline{/-141.8^\circ}\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/-240^\circ} = 6.81\underline{/-261.8^\circ}\ \text{A} = 6.81\underline{/98.2^\circ}\ \text{A}$$

PRACTICE PROBLEM 12.2

A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\mathbf{V}_{an} = 120\underline{/30^\circ}$ V, find: (a) the line voltages, (b) the line currents.

Answer: (a) $207.85\angle 60^\circ$ V, $207.85\angle -60^\circ$ V, $207.85\angle -180^\circ$ V, (b) $3.75\angle -8.66^\circ$ A, $3.75\angle -128.66^\circ$ A, $3.75\angle -248.66^\circ$ A.

12.4 BALANCED WYE-DELTA CONNECTION

> A balanced Y-Δ system consists of a balanced Y-connected source feeding a balanced Δ-connected load.

This is perhaps the most practical three-phase system, as the three-phase sources are usually Y-connected while the three-phase loads are usually Δ-connected.

The balanced Y-delta system is shown in Fig. 12.14, where the source is wye-connected and the load is Δ-connected. There is, of course, no neutral connection from source to load for this case. Assuming the positive sequence, the phase voltages are again

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ \tag{12.19}$$

As shown in Section 12.3, the line voltages are

$$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \sqrt{3}V_p\angle -90^\circ = \mathbf{V}_{BC}$$
$$\mathbf{V}_{ca} = \sqrt{3}V_p\angle -210^\circ = \mathbf{V}_{CA} \tag{12.20}$$

showing that the line voltages are equal to the voltages across the load impedances for this system configuration. From these voltages, we can obtain the phase currents as

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} \tag{12.21}$$

These currents have the same magnitude but are out of phase with each other by 120°.

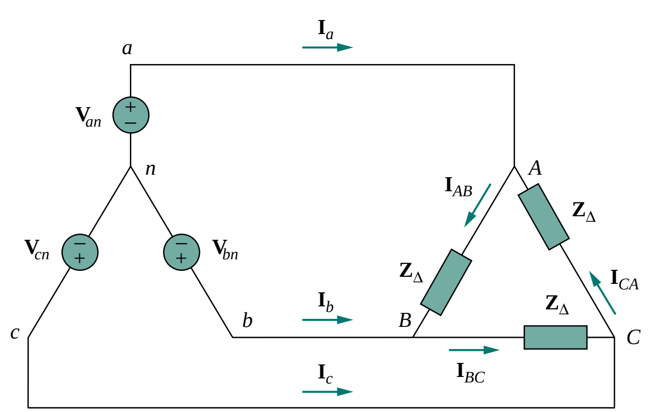

Figure 12.14 Balanced Y-Δ connection.

Another way to get these phase currents is to apply KVL. For example, applying KVL around loop $aABbna$ gives

$$-\mathbf{V}_{an} + \mathbf{Z}_\Delta \mathbf{I}_{AB} + \mathbf{V}_{bn} = 0$$

or

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{an} - \mathbf{V}_{bn}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} \tag{12.22}$$

which is the same as Eq. (12.21). This is the more general way of finding the phase currents.

The line currents are obtained from the phase currents by applying KCL at nodes A, B, and C. Thus,

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC} \tag{12.23}$$

Since $\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/-240^\circ}$,

$$\begin{aligned} \mathbf{I}_a &= \mathbf{I}_{AB} - \mathbf{I}_{CA} = \mathbf{I}_{AB}(1 - 1\underline{/-240^\circ}) \\ &= \mathbf{I}_{AB}(1 + 0.5 - j0.866) = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} \end{aligned} \tag{12.24}$$

showing that the magnitude I_L of the line current is $\sqrt{3}$ times the magnitude I_p of the phase current, or

$$\boxed{I_L = \sqrt{3} I_p} \tag{12.25}$$

where

$$I_L = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c| \tag{12.26}$$

and

$$I_p = |\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}| \tag{12.27}$$

Also, the line currents lag the corresponding phase currents by 30°, assuming the positive sequence. Figure 12.15 is a phasor diagram illustrating the relationship between the phase and line currents.

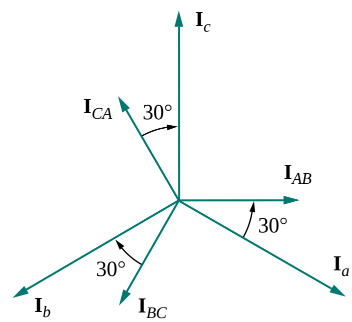

Figure 12.15 Phasor diagram illustrating the relationship between phase and line currents.

An alternative way of analyzing the Y-Δ circuit is to transform the Δ-connected load to an equivalent Y-connected load. Using the Δ-Y transformation formula in Eq. (9.69),

$$\boxed{\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3}} \tag{12.28}$$

After this transformation, we now have a Y-Y system as in Fig. 12.10. The three-phase Y-Δ system in Fig. 12.14 can be replaced by the single-phase equivalent circuit in Fig. 12.16. This allows us to calculate only the line currents. The phase currents are obtained using Eq. (12.25) and utilizing the fact that each of the phase currents leads the corresponding line current by 30°.

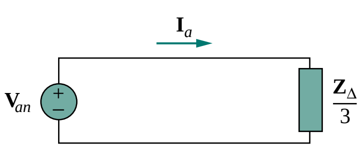

Figure 12.16 A single-phase equivalent circuit of a balanced Y-Δ circuit.

EXAMPLE 12.3

A balanced abc-sequence Y-connected source with $\mathbf{V}_{an} = 100\underline{/10^\circ}$ V is connected to a Δ-connected balanced load $(8 + j4)\ \Omega$ per phase. Calculate the phase and line currents.

Solution:

This can be solved in two ways.

METHOD 1 The load impedance is

$$\mathbf{Z}_\Delta = 8 + j4 = 8.944\underline{/26.57^\circ}\ \Omega$$

If the phase voltage $\mathbf{V}_{an} = 100\underline{/10^\circ}$, then the line voltage is

$$\mathbf{V}_{ab} = \mathbf{V}_{an}\sqrt{3}\underline{/30^\circ} = 100\sqrt{3}\underline{/10^\circ + 30^\circ} = \mathbf{V}_{AB}$$

or

$$\mathbf{V}_{AB} = 173.2\underline{/40^\circ}\ \text{V}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{173.2\underline{/40^\circ}}{8.944\underline{/26.57^\circ}} = 19.36\underline{/13.43^\circ}\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\underline{/-120^\circ} = 19.36\underline{/-106.57^\circ}\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/+120^\circ} = 19.36\underline{/133.43^\circ}\ \text{A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} = \sqrt{3}(19.36)\underline{/13.43^\circ - 30^\circ}$$

$$= 33.53\underline{/-16.57^\circ}\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 33.53\underline{/-136.57^\circ}\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/+120^\circ} = 33.53\underline{/103.43^\circ}\ \text{A}$$

METHOD 2 Alternatively, using single-phase analysis,

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_\Delta/3} = \frac{100\underline{/10^\circ}}{2.981\underline{/26.57^\circ}} = 33.54\underline{/-16.57^\circ}\ \text{A}$$

as above. Other line currents are obtained using the *abc* phase sequence.

PRACTICE PROBLEM 12.3

One line voltage of a balanced Y-connected source is $\mathbf{V}_{AB} = 180\underline{/-20^\circ}$ V. If the source is connected to a Δ-connected load of $20\underline{/40^\circ}\ \Omega$, find the phase and line currents. Assume the *abc* sequence.

Answer: $9\underline{/-60^\circ}$, $9\underline{/-180^\circ}$, $9\underline{/60^\circ}$, $15.59\underline{/-90^\circ}$, $15.59\underline{/-210^\circ}$, $15.59\underline{/30^\circ}$ A.

12.5 BALANCED DELTA-DELTA CONNECTION

A balanced Δ-Δ system is one in which both the balanced source and balanced load are Δ-connected.

The source as well as the load may be delta-connected as shown in Fig. 12.17. Our goal is to obtain the phase and line currents as usual. Assuming a positive sequence, the phase voltages for a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bc} = V_p\angle -120^\circ, \qquad \mathbf{V}_{ca} = V_p\angle +120^\circ \tag{12.29}$$

The line voltages are the same as the phase voltages. From Fig. 12.17, assuming there is no line impedances, the phase voltages of the delta-connected source are equal to the voltages across the impedances; that is,

$$\mathbf{V}_{ab} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \mathbf{V}_{BC}, \qquad \mathbf{V}_{ca} = \mathbf{V}_{CA} \tag{12.30}$$

Hence, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_\Delta}$$
$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_\Delta} \tag{12.31}$$

Since the load is delta-connected just as in the previous section, some of the formulas derived there apply here. The line currents are obtained from the phase currents by applying KCL at nodes A, B, and C, as we did in the previous section:

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC} \tag{12.32}$$

Also, as shown in the last section, each line current lags the corresponding phase current by 30°; the magnitude I_L of the line current is $\sqrt{3}$ times the magnitude I_p of the phase current,

$$I_L = \sqrt{3} I_p \tag{12.33}$$

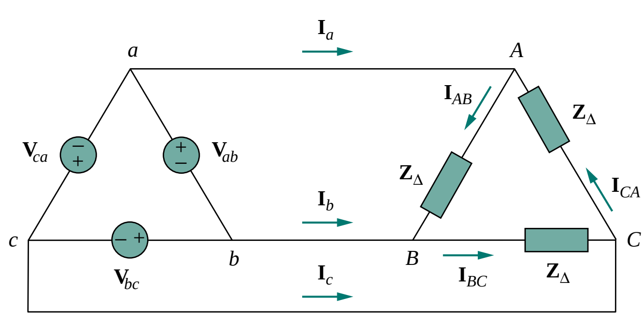

Figure 12.17 A balanced Δ-Δ connection.

An alternative way of analyzing the Δ-Δ circuit is to convert both the source and the load to their Y equivalents. We already know that $\mathbf{Z}_Y = \mathbf{Z}_\Delta/3$. To convert a Δ-connected source to a Y-connected source, see the next section.

EXAMPLE 12.4

A balanced Δ-connected load having an impedance $20 - j15\ \Omega$ is connected to a Δ-connected, positive-sequence generator having $\mathbf{V}_{ab} = 330\angle 0^\circ$ V. Calculate the phase currents of the load and the line currents.

Solution:

The load impedance per phase is

$$\mathbf{Z}_\Delta = 20 - j15 = 25\angle -36.87^\circ\ \Omega$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{330\angle 0^\circ}{25\angle -36.87} = 13.2\angle 36.87^\circ \text{ A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\angle -120^\circ = 13.2\angle -83.13^\circ \text{ A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle +120^\circ = 13.2\angle 156.87^\circ \text{ A}$$

For a delta load, the line current always lags the corresponding phase current by 30° and has a magnitude $\sqrt{3}$ times that of the phase current. Hence, the line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ = (13.2\angle 36.87^\circ)(\sqrt{3}\angle -30^\circ)$$
$$= 22.86\angle 6.87^\circ \text{ A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 22.86\angle -113.13^\circ \text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 22.86\angle 126.87^\circ \text{ A}$$

PRACTICE PROBLEM 12.4

A positive-sequence, balanced Δ-connected source supplies a balanced Δ-connected load. If the impedance per phase of the load is $18 + j12\ \Omega$ and $\mathbf{I}_a = 22.5\angle 35^\circ$ A, find $\mathbf{I}_{AB}$ and $\mathbf{V}_{AB}$.

Answer: $13\angle 65^\circ$ A, $281.2\angle 98.69^\circ$ V.

12.6 BALANCED DELTA-WYE CONNECTION

A balanced Δ-Y system consists of a balanced Δ-connected source feeding a balanced Y-connected load.

Consider the Δ-Y circuit in Fig. 12.18. Again, assuming the *abc* sequence, the phase voltages of a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ, \qquad \mathbf{V}_{bc} = V_p\angle -120^\circ$$
$$\mathbf{V}_{ca} = V_p\angle +120^\circ \tag{12.34}$$

These are also the line voltages as well as the phase voltages.

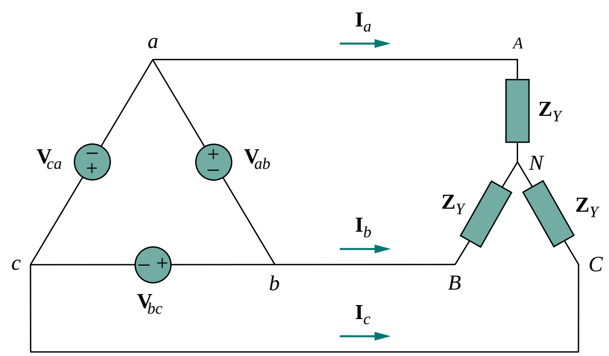

Figure 12.18 A balanced Δ-Y connection.

We can obtain the line currents in many ways. One way is to apply KVL to loop $aANBba$ in Fig. 12.18, writing

$$-\mathbf{V}_{ab} + \mathbf{Z}_Y\mathbf{I}_a - \mathbf{Z}_Y\mathbf{I}_b = 0$$

or

$$\mathbf{Z}_Y(\mathbf{I}_a - \mathbf{I}_b) = \mathbf{V}_{ab} = V_p\underline{/0^\circ}$$

Thus,

$$\mathbf{I}_a - \mathbf{I}_b = \frac{V_p\underline{/0^\circ}}{\mathbf{Z}_Y} \tag{12.35}$$

But $\mathbf{I}_b$ lags $\mathbf{I}_a$ by 120°, since we assumed the abc sequence; that is, $\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ}$. Hence,

$$\begin{aligned}\mathbf{I}_a - \mathbf{I}_b &= \mathbf{I}_a(1 - 1\underline{/-120^\circ}) \\ &= \mathbf{I}_a\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \mathbf{I}_a\sqrt{3}\underline{/30^\circ}\end{aligned} \tag{12.36}$$

Substituting Eq. (12.36) into Eq. (12.35) gives

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\underline{/-30^\circ}}{\mathbf{Z}_Y} \tag{12.37}$$

From this, we obtain the other line currents $\mathbf{I}_b$ and $\mathbf{I}_c$ using the positive phase sequence, i.e., $\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ}$, $\mathbf{I}_c = \mathbf{I}_a\underline{/+120^\circ}$. The phase currents are equal to the line currents.

Another way to obtain the line currents is to replace the delta-connected source with its equivalent wye-connected source, as shown in Fig. 12.19. In Section 12.3, we found that the line-to-line voltages of a wye-connected source lead their corresponding phase voltages by 30°. Therefore, we obtain each phase voltage of the equivalent wye-connected source by dividing the corresponding line voltage of the delta-connected source by $\sqrt{3}$ and shifting its phase by −30°. Thus, the equivalent wye-connected source has the phase voltages

$$\mathbf{V}_{an} = \frac{V_p}{\sqrt{3}}\underline{/-30^\circ}$$
$$\mathbf{V}_{bn} = \frac{V_p}{\sqrt{3}}\underline{/-150^\circ}, \qquad \mathbf{V}_{cn} = \frac{V_p}{\sqrt{3}}\underline{/+90^\circ} \tag{12.38}$$

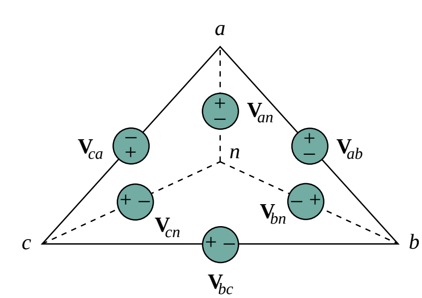

Figure 12.19 Transforming a Δ-connected source to an equivalent Y-connected source.

If the delta-connected source has source impedance $\mathbf{Z}_s$ per phase, the equivalent wye-connected source will have a source impedance of $\mathbf{Z}_s/3$ per phase, according to Eq. (9.69).

Once the source is transformed to wye, the circuit becomes a wye-wye system. Therefore, we can use the equivalent single-phase circuit shown in Fig. 12.20, from which the line current for phase a is

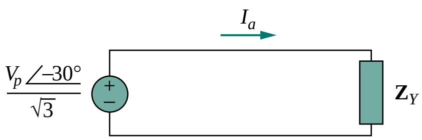

Figure 12.20 The single-phase equivalent circuit.

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\,\angle -30^\circ}{\mathbf{Z}_Y} \tag{12.39}$$

which is the same as Eq. (12.37).

Alternatively, we may transform the wye-connected load to an equivalent delta-connected load. This results in a delta-delta system, which can be analyzed as in Section 12.5. Note that

$$\mathbf{V}_{AN} = \mathbf{I}_a\mathbf{Z}_Y = \frac{V_p}{\sqrt{3}}\angle -30^\circ \tag{12.40}$$

$$\mathbf{V}_{BN} = \mathbf{V}_{AN}\angle -120^\circ, \qquad \mathbf{V}_{CN} = \mathbf{V}_{AN}\angle +120^\circ$$

As stated earlier, the delta-connected load is more desirable than the wye-connected load. It is easier to alter the loads in any one phase of the delta-connected loads, as the individual loads are connected directly across the lines. However, the delta-connected source is hardly used in practice, because any slight imbalance in the phase voltages will result in unwanted circulating currents.

Table 12.1 presents a summary of the formulas for phase currents and voltages and line currents and voltages for the four connections. Students are advised not to memorize the formulas but to understand how they are derived. The formulas can always be obtained by directly applying KCL and KVL to the appropriate three-phase circuits.

TABLE 12.1 Summary of phase and line voltages/currents for balanced three-phase systems[1].

Connection	Phase voltages/currents	Line voltages/currents
Y-Y	$\mathbf{V}_{an} = V_p\angle 0^\circ$	$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ$
	$\mathbf{V}_{bn} = V_p\angle -120^\circ$	$\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle -120^\circ$
	$\mathbf{V}_{cn} = V_p\angle +120^\circ$	$\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle +120^\circ$
	Same as line currents	$\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$
		$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$
		$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Y-Δ	$\mathbf{V}_{an} = V_p\angle 0^\circ$	$\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3}V_p\angle 30^\circ$
	$\mathbf{V}_{bn} = V_p\angle -120^\circ$	$\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab}\angle -120^\circ$
	$\mathbf{V}_{cn} = V_p\angle +120^\circ$	$\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab}\angle +120^\circ$
	$\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$
	$\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$	$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$
	$\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$	$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

[1] *Positive or abc sequence is assumed.*

TABLE 12.1 (*continued*)

Connection	Phase voltages/currents	Line voltages/currents
Δ-Δ	$\mathbf{V}_{ab} = V_p\angle 0^\circ$	Same as phase voltages
	$\mathbf{V}_{bc} = V_p\angle -120^\circ$	
	$\mathbf{V}_{ca} = V_p\angle +120^\circ$	
	$\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$
	$\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$	$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$
	$\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$	$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Δ-Y	$\mathbf{V}_{ab} = V_p\angle 0^\circ$	Same as phase voltages
	$\mathbf{V}_{bc} = V_p\angle -120^\circ$	
	$\mathbf{V}_{ca} = V_p\angle +120^\circ$	
	Same as line currents	$\mathbf{I}_a = \dfrac{V_p\angle -30^\circ}{\sqrt{3}\mathbf{Z}_Y}$
		$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$
		$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

EXAMPLE 12.5

A balanced Y-connected load with a phase resistance of 40 Ω and a reactance of 25 Ω is supplied by a balanced, positive sequence Δ-connected source with a line voltage of 210 V. Calculate the phase currents. Use $\mathbf{V}_{ab}$ as reference.

Solution:

The load impedance is

$$\mathbf{Z}_Y = 40 + j25 = 47.17\angle 32^\circ\ \Omega$$

and the source voltage is

$$\mathbf{V}_{ab} = 210\angle 0^\circ\ \text{V}$$

When the Δ-connected source is transformed to a Y-connected source,

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}}\angle -30^\circ = 121.2\angle -30^\circ\ \text{V}$$

The line currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{121.2\angle -30^\circ}{47.12\angle 32^\circ} = 2.57\angle -62^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 2.57\angle -182^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle 120^\circ = 2.57\angle 58^\circ\ \text{A}$$

which are the same as the phase currents.

PRACTICE PROBLEM 12.5

In a balanced Δ-Y circuit, $\mathbf{V}_{ab} = 240\angle 15^\circ$ and $\mathbf{Z}_Y = (12 + j15)\ \Omega$. Calculate the line currents.

Answer: $7.21\angle -66.34^\circ$, $7.21\angle -186.34^\circ$, $7.21\angle 53.66^\circ$ A.

12.7 POWER IN A BALANCED SYSTEM

Let us now consider the power in a balanced three-phase system. We begin by examining the instantaneous power absorbed by the load. This requires that the analysis be done in the time domain. For a Y-connected load, the phase voltages are

$$v_{AN} = \sqrt{2}V_p \cos\omega t, \qquad v_{BN} = \sqrt{2}V_p \cos(\omega t - 120^\circ)$$
$$v_{CN} = \sqrt{2}V_p \cos(\omega t + 120^\circ) \tag{12.41}$$

where the factor $\sqrt{2}$ is necessary because V_p has been defined as the rms value of the phase voltage. If $\mathbf{Z}_Y = Z\angle\theta$, the phase currents lag behind their corresponding phase voltages by θ. Thus,

$$i_a = \sqrt{2}I_p \cos(\omega t - \theta), \qquad i_b = \sqrt{2}I_p \cos(\omega t - \theta - 120^\circ)$$
$$i_c = \sqrt{2}I_p \cos(\omega t - \theta + 120^\circ) \tag{12.42}$$

where I_p is the rms value of the phase current. The total instantaneous power in the load is the sum of the instantaneous powers in the three phases; that is,

$$\begin{aligned} p &= p_a + p_b + p_c = v_{AN}i_a + v_{BN}i_b + v_{CN}i_c \\ &= 2V_pI_p[\cos\omega t\cos(\omega t - \theta) \\ &\qquad + \cos(\omega t - 120^\circ)\cos(\omega t - \theta - 120^\circ) \\ &\qquad + \cos(\omega t + 120^\circ)\cos(\omega t - \theta + 120^\circ)] \end{aligned} \tag{12.43}$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)] \tag{12.44}$$

gives

$$\begin{aligned} p &= V_pI_p[3\cos\theta + \cos(2\omega t - \theta) + \cos(2\omega t - \theta - 240^\circ) \\ &\qquad + \cos(2\omega t - \theta + 240^\circ)] \\ &= V_pI_p[3\cos\theta + \cos\alpha + \cos\alpha\cos 240^\circ + \sin\alpha\sin 240^\circ \\ &\qquad + \cos\alpha\cos 240^\circ - \sin\alpha\sin 240^\circ] \\ &\qquad \text{where } \alpha = 2\omega t - \theta \\ &= V_pI_p\left[3\cos\theta + \cos\alpha + 2\left(-\frac{1}{2}\right)\cos\alpha\right] = 3V_pI_p\cos\theta \end{aligned} \tag{12.45}$$

Thus the total instantaneous power in a balanced three-phase system is constant—it does not change with time as the instantaneous power of each phase does. This result is true whether the load is Y- or Δ-connected.

This is one important reason for using a three-phase system to generate and distribute power. We will look into another reason a little later.

Since the total instantaneous power is independent of time, the average power per phase P_p for either the Δ-connected load or the Y-connected load is $p/3$, or

$$P_p = V_p I_p \cos\theta \tag{12.46}$$

and the reactive power per phase is

$$Q_p = V_p I_p \sin\theta \tag{12.47}$$

The apparent power per phase is

$$S_p = V_p I_p \tag{12.48}$$

The complex power per phase is

$$\mathbf{S}_p = P_p + jQ_p = \mathbf{V}_p \mathbf{I}_p^* \tag{12.49}$$

where $\mathbf{V}_p$ and $\mathbf{I}_p$ are the phase voltage and phase current with magnitudes V_p and I_p, respectively. The total average power is the sum of the average powers in the phases:

$$P = P_a + P_b + P_c = 3P_p = 3V_p I_p \cos\theta = \sqrt{3} V_L I_L \cos\theta \tag{12.50}$$

For a Y-connected load, $I_L = I_p$ but $V_L = \sqrt{3}V_p$, whereas for a Δ-connected load, $I_L = \sqrt{3}I_p$ but $V_L = V_p$. Thus, Eq. (12.50) applies for both Y-connected and Δ-connected loads. Similarly, the total reactive power is

$$Q = 3V_p I_p \sin\theta = 3Q_p = \sqrt{3} V_L I_L \sin\theta \tag{12.51}$$

and the total complex power is

$$\boxed{\mathbf{S} = 3\mathbf{S}_p = 3\mathbf{V}_p \mathbf{I}_p^* = 3I_p^2 \mathbf{Z}_p = \frac{3V_p^2}{\mathbf{Z}_p^*}} \tag{12.52}$$

where $\mathbf{Z}_p = Z_p \angle\theta$ is the load impedance per phase. ($\mathbf{Z}_p$ could be $\mathbf{Z}_Y$ or $\mathbf{Z}_\Delta$.) Alternatively, we may write Eq. (12.52) as

$$\boxed{\mathbf{S} = P + jQ = \sqrt{3} V_L I_L \angle\theta} \tag{12.53}$$

Remember that V_p, I_p, V_L, and I_L are all rms values and that θ is the angle of the load impedance or the angle between the phase voltage and the phase current.

A second major advantage of three-phase systems for power distribution is that the three-phase system uses a lesser amount of wire than the single-phase system for the same line voltage V_L and the same absorbed power P_L. We will compare these cases and assume in both that the wires are of the same material (e.g., copper with resistivity ρ), of the same length ℓ, and that the loads are resistive (i.e., unity power factor). For the two-wire single-phase system in Fig. 12.21(a), $I_L = P_L/V_L$, so the power loss in the two wires is

$$P_{\text{loss}} = 2I_L^2 R = 2R\frac{P_L^2}{V_L^2} \tag{12.54}$$

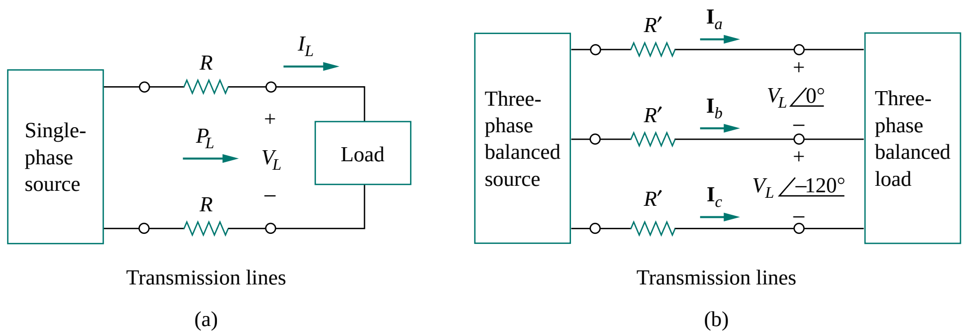

Figure 12.21 Comparing the power loss in (a) a single-phase system, and (b) a three-phase system.

For the three-wire three-phase system in Fig. 12.21(b), $I'_L = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c| = P_L/\sqrt{3}V_L$ from Eq. (12.50). The power loss in the three wires is

$$P'_{\text{loss}} = 3(I'_L)^2R' = 3R'\frac{P_L^2}{3V_L^2} = R'\frac{P_L^2}{V_L^2} \tag{12.55}$$

Equations (12.54) and (12.55) show that for the same total power delivered P_L and same line voltage V_L,

$$\frac{P_{\text{loss}}}{P'_{\text{loss}}} = \frac{2R}{R'} \tag{12.56}$$

But from Chapter 2, $R = \rho\ell/\pi r^2$ and $R' = \rho\ell/\pi r'^2$, where r and r' are the radii of the wires. Thus,

$$\frac{P_{\text{loss}}}{P'_{\text{loss}}} = \frac{2r'^2}{r^2} \tag{12.57}$$

If the same power loss is tolerated in both systems, then $r^2 = 2r'^2$. The ratio of material required is determined by the number of wires and their volumes, so

$$\begin{aligned}\frac{\text{Material for single-phase}}{\text{Material for three-phase}} &= \frac{2(\pi r^2\ell)}{3(\pi r'^2\ell)} = \frac{2r^2}{3r'^2}\\ &= \frac{2}{3}(2) = 1.333\end{aligned} \tag{12.58}$$

since $r^2 = 2r'^2$. Equation (12.58) shows that the single-phase system uses 33 percent more material than the three-phase system or that the three-phase system uses only 75 percent of the material used in the equivalent single-phase system. In other words, considerably less material is needed to deliver the same power with a three-phase system than is required for a single-phase system.

EXAMPLE 12.6

Refer to the circuit in Fig. 12.13 (in Example 12.2). Determine the total average power, reactive power, and complex power at the source and at the load.

Solution:

It is sufficient to consider one phase, as the system is balanced. For phase a,

$$\mathbf{V}_p = 110\underline{/0^\circ} \text{ V} \qquad \text{and} \qquad \mathbf{I}_p = 6.81\underline{/-21.8^\circ} \text{ A}$$

Thus, at the source, the complex power supplied is

$$\begin{aligned}\mathbf{S}_s &= -3\mathbf{V}_p\mathbf{I}_p^* = 3(110\underline{/0^\circ})(6.81\underline{/21.8^\circ}) \\ &= -2247\underline{/21.8^\circ} = -(2087 + j834.6) \text{ VA}\end{aligned}$$

The real or average power supplied is −2087 W and the reactive power is −834.6 VAR.

At the load, the complex power absorbed is

$$\mathbf{S}_L = 3|\mathbf{I}_p|^2\mathbf{Z}_p$$

where $\mathbf{Z}_p = 10 + j8 = 12.81\underline{/38.66^\circ}$ and $\mathbf{I}_p = \mathbf{I}_a = 6.81\underline{/-21.8^\circ}$. Hence

$$\begin{aligned}\mathbf{S}_L &= 3(6.81)^2 12.81\underline{/38.66^\circ} = 1782\underline{/38.66} \\ &= (1392 + j1113) \text{ VA}\end{aligned}$$

The real power absorbed is 1391.7 W and the reactive power absorbed is 1113.3 VAR. The difference between the two complex powers is absorbed by the line impedance $(5 - j2)\ \Omega$. To show that this is the case, we find the complex power absorbed by the line as

$$\mathbf{S}_\ell = 3|\mathbf{I}_p|^2\mathbf{Z}_\ell = 3(6.81)^2(5 - j2) = 695.6 - j278.3 \text{ VA}$$

which is the difference between $\mathbf{S}_s$ and $\mathbf{S}_L$, that is, $\mathbf{S}_s + \mathbf{S}_\ell + \mathbf{S}_L = 0$, as expected.

PRACTICE PROBLEM 12.6

For the Y-Y circuit in Practice Prob. 12.2, calculate the complex power at the source and at the load.

Answer: $(1054 + j843.3)$ VA, $(1012 + j801.6)$ VA.

EXAMPLE 12.7

A three-phase motor can be regarded as a balanced Y-load. A three-phase motor draws 5.6 kW when the line voltage is 220 V and the line current is 18.2 A. Determine the power factor of the motor.

Solution:

The apparent power is

$$S = \sqrt{3}V_L I_L = \sqrt{3}(220)(18.2) = 6935.13 \text{ VA}$$

Since the real power is

$$P = S\cos\theta = 5600 \text{ W}$$

the power factor is

$$\text{pf} = \cos\theta = \frac{P}{S} = \frac{5600}{6935.13} = 0.8075$$

PRACTICE PROBLEM 12.7

Calculate the line current required for a 30-kW three-phase motor having a power factor of 0.85 lagging if it is connected to a balanced source with a line voltage of 440 V.

Answer: 50.94 A.

EXAMPLE 12.8

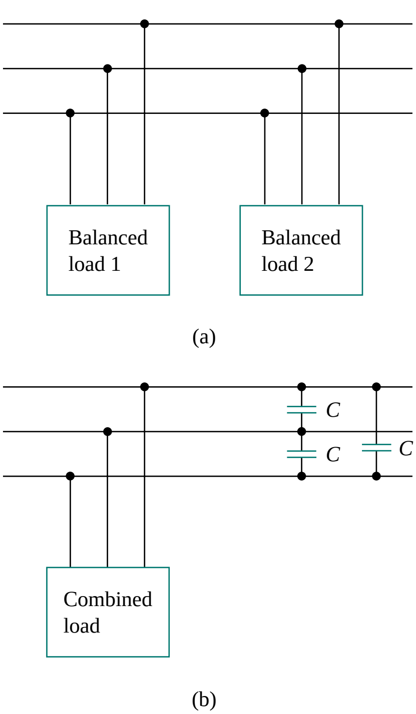

Figure 12.22 For Example 12.8: (a) The original balanced loads, (b) the combined load with improved power factor.

Two balanced loads are connected to a 240-kV rms 60-Hz line, as shown in Fig. 12.22(a). Load 1 draws 30 kW at a power factor of 0.6 lagging, while load 2 draws 45 kVAR at a power factor of 0.8 lagging. Assuming the *abc* sequence, determine: (a) the complex, real, and reactive powers absorbed by the combined load, (b) the line currents, and (c) the kVAR rating of the three capacitors Δ-connected in parallel with the load that will raise the power factor to 0.9 lagging and the capacitance of each capacitor.

Solution:

(a) For load 1, given that $P_1 = 30$ kW and $\cos\theta_1 = 0.6$, then $\sin\theta_1 = 0.8$. Hence,

$$S_1 = \frac{P_1}{\cos\theta_1} = \frac{30\text{ kW}}{0.6} = 50\text{ kVA}$$

and $Q_1 = S_1 \sin\theta_1 = 50(0.8) = 40$ kVAR. Thus, the complex power due to load 1 is

$$\mathbf{S}_1 = P_1 + jQ_1 = 30 + j40\text{ kVA} \tag{12.8.1}$$

For load 2, if $Q_2 = 45$ kVAR and $\cos\theta_2 = 0.8$, then $\sin\theta_2 = 0.6$. We find

$$S_2 = \frac{Q_2}{\sin\theta_2} = \frac{45\text{ kVA}}{0.6} = 75\text{ kVA}$$

and $P_2 = S_2\cos\theta_2 = 75(0.8) = 60$ kW. Therefore the complex power due to load 2 is

$$\mathbf{S}_2 = P_2 + jQ_2 = 60 + j45\text{ kVA} \tag{12.8.2}$$

From Eqs. (12.8.1) and (12.8.2), the total complex power absorbed by the load is

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = 90 + j85\text{ kVA} = 123.8\underline{/43.36^\circ}\text{ kVA} \tag{12.8.3}$$

which has a power factor of $\cos 43.36^\circ = 0.727$ lagging. The real power is then 90 kW, while the reactive power is 85 kVAR.

(b) Since $S = \sqrt{3}V_L I_L$, the line current is

$$I_L = \frac{S}{\sqrt{3}V_L} \tag{12.8.4}$$

We apply this to each load, keeping in mind that for both loads, $V_L = 240$ kV. For load 1,

$$I_{L1} = \frac{50{,}000}{\sqrt{3}\,240{,}000} = 120.28 \text{ mA}$$

Since the power factor is lagging, the line current lags the line voltage by $\theta_1 = \cos^{-1} 0.6 = 53.13^\circ$. Thus,

$$\mathbf{I}_{a1} = 120.28\underline{/-53.13^\circ}$$

For load 2,

$$I_{L2} = \frac{75{,}000}{\sqrt{3}\,240{,}000} = 180.42 \text{ mA}$$

and the line current lags the line voltage by $\theta_2 = \cos^{-1} 0.8 = 36.87^\circ$. Hence,

$$\mathbf{I}_{a2} = 180.42\underline{/-36.87^\circ}$$

The total line current is

$$\begin{aligned}\mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} &= 120.28\underline{/-53.13^\circ} + 180.42\underline{/-36.87^\circ}\\ &= (72.168 - j96.224) + (144.336 - j108.252)\\ &= 216.5 - j204.472 = 297.8\underline{/-43.36^\circ} \text{ mA}\end{aligned}$$

Alternatively, we could obtain the current from the total complex power using Eq. (12.8.4) as

$$I_L = \frac{123{,}800}{\sqrt{3}\,240{,}000} = 297.82 \text{ mA}$$

and

$$\mathbf{I}_a = 297.82\underline{/-43.36^\circ} \text{ mA}$$

which is the same as before. The other line currents, $\mathbf{I}_{b2}$ and $\mathbf{I}_{ca}$, can be obtained according to the abc sequence (i.e., $\mathbf{I}_b = 297.82\underline{/-163.36^\circ}$ mA and $\mathbf{I}_c = 297.82\underline{/76.64^\circ}$ mA).

(c) We can find the reactive power needed to bring the power factor to 0.9 lagging using Eq. (11.59),

$$Q_C = P(\tan\theta_{\text{old}} - \tan\theta_{\text{new}})$$

where $P = 90$ kW, $\theta_{\text{old}} = 43.36^\circ$, and $\theta_{\text{new}} = \cos^{-1} 0.9 = 25.84^\circ$. Hence,

$$Q_C = 90{,}000(\tan 43.36^\circ - \tan 25.04^\circ) = 41.4 \text{ kVAR}$$

This reactive power is for the three capacitors. For each capacitor, the rating $Q'_C = 13.8$ kVAR. From Eq. (11.60), the required capacitance is

$$C = \frac{Q'_C}{\omega V_{\text{rms}}^2}$$

Since the capacitors are Δ-connected as shown in Fig. 12.22(b), V_{rms} in the above formula is the line-to-line or line voltage, which is 240 kV. Thus,

$$C = \frac{13{,}800}{(2\pi 60)(240{,}000)^2} = 635.5 \text{ pF}$$